RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2611

Mechanics 5

Tuesday **29 JUNE 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take g = 9.8 m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 A ball of mass m is projected from horizontal ground with initial velocity $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upward unit vectors respectively. The ball is subject to a resistance to motion of $-mk\mathbf{v}$ where k is a constant and $\mathbf{v}$ is the velocity of the ball at time t after projection.

(i) Write down the vector equation of motion. Hence write down the equations of motion in the $\mathbf{i}$ and $\mathbf{j}$ directions. [4]

(ii) Solve the equations to show that the horizontal and vertical components of velocity of the ball are

$$\dot{x} = u_1 e^{-kt},$$
$$\dot{y} = \frac{1}{k}\left((g + ku_2)e^{-kt} - g\right).$$ [6]

(iii) Hence show that the maximum height reached by the ball is $\dfrac{u_2}{k} - \dfrac{g}{k^2}\ln\left(1 + \dfrac{ku_2}{g}\right)$.

Use the approximation $\ln(1 + x) \approx x - \frac{1}{2}x^2$ for small x to show that this is approximately the same as the corresponding result for unresisted motion. [10]

2 A section of a straight river of constant width 50 m flows at a constant speed of $V \text{ m s}^{-1}$. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel and perpendicular to the river as shown in Fig. 2. A small boat travels at 8 m s^{-1} relative to the water.

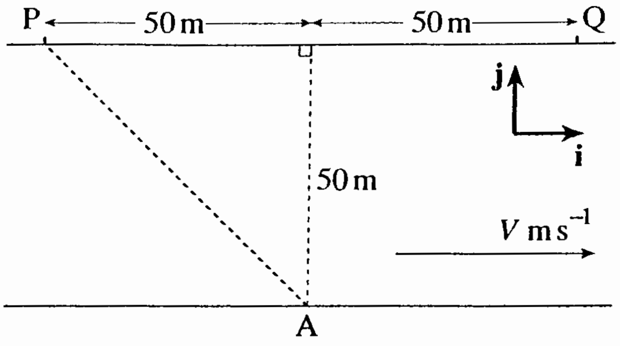


Fig. 2

(i) The boat sets off from a point A on one bank in order to reach a point P on the opposite bank, 50 m upstream from A. The boat sets a course at an angle of 30° to the bank and travels directly to P.

Calculate V and the resultant speed of the boat. [6]

(ii) On another occasion, the boat travels from A directly to a point Q, on the opposite bank 50 m downstream of A. Calculate the angle with the bank of the course that must be set. [3]

On a third occasion the boat travels directly from A to P. A barge is travelling downstream in the middle of the river. At the moment when the boat starts from A the front of the barge is 100 m upstream from A. The barge travels at a constant speed of $U \text{ m s}^{-1}$ relative to the water.

(iii) Show that the velocity of the boat relative to the front of the barge is $-(U + 4\sqrt{3})\mathbf{i} + 4\mathbf{j}$. Find the value of U for which the boat collides with the front of the barge. [4]

(iv) Show that the distance, d m, between the boat and the front of the barge at time t seconds after the boat starts from A is given by

$$d^2 = (a^2 + 16)t^2 - 200(a + 1)t + 10\,625,$$

where $a = U + 4\sqrt{3}$. Find the range of values of U for which d is always greater than 5. (You should give your answers to 3 significant figures). [7]

[Turn over

3 A particle is moving in a plane. Unit vectors in the radial and transverse directions are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ respectively.

(i) Using the results $\dfrac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $\dfrac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot{\theta}\,\hat{\mathbf{r}}$ derive an expression for the velocity of the particle and show that the acceleration is $(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\hat{\boldsymbol{\theta}}$.

A satellite of mass m orbiting a planet at the origin is subject to a force $-\dfrac{mk\mathbf{r}}{r^3}$ where k is a positive constant. [7]

(ii) Show that $r^2\dot{\theta}$ is constant. [3]

(iii) Denoting the constant value of $r^2\dot{\theta}$ by h, find $\ddot{r}$ in terms of r, k and h and hence show that $\dot{r}^2 = \dfrac{2kr - h^2}{r^2} + A$, where A is an arbitrary constant. [6]

(iv) Hence find the speed of the satellite in terms of r, k and A. Hence show that, if the orbit is not circular, the maximum speed of the satellite occurs when it is closest to the planet. [4]

4 A circular disc of radius a has mass M. Its mass per unit area is proportional to the distance from the circumference of the disc.

(i) Show that the moment of inertia of the disc about an axis through its centre and perpendicular to the disc is $\frac{3}{10}Ma^2$. [8]

(ii) Find the moment of inertia of the disc about a diameter and hence show that the moment of inertia about a tangent is $\frac{23}{20}Ma^2$. [4]

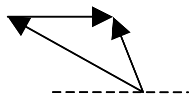
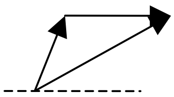
The disc is suspended so that it can swing freely about a horizontal tangent.

(iii) The disc is held horizontally and then released. Find its angular speed when it passes through the vertical position. [4]

(iv) In a new situation, the disc performs small oscillations about its equilibrium position. Find the period of these oscillations. [4]

Mark Scheme

1(i)	$m\dfrac{d\mathbf{v}}{dt} = -mg\mathbf{j} - mk\mathbf{v}$	M1 A1	N2L (3 terms)
	$\dfrac{d\dot{x}}{dt} = -k\dot{x}, \quad \dfrac{d\dot{y}}{dt} = -g - k\dot{y}$	B1 B1	
			4
(ii)	$\dot{x} = Ae^{-kt}$	M1	
	$t = 0, \dot{x} = u_1 \Rightarrow \dot{x} = u_1 e^{-kt}$	E1	
	$\int \dfrac{d\dot{y}}{-g - k\dot{y}} = \int dt$	M1	separate variables
	$-\dfrac{1}{k}\ln\lvert g + k\dot{y}\rvert = t + c_1$	M1	integrate
	$g + k\dot{y} = B\,e^{-kt}$	M1	rearrange and use conditions
	$t = 0, \dot{y} = u_2 \Rightarrow \dot{y} = \dfrac{1}{k}((g + ku_2)e^{-kt} - g)$	E1	
			6
(iii)	$\dot{y} = 0 \Rightarrow e^{-kt} = \dfrac{g}{g + ku_2}$	M1	solve for time at max. height
	$t = -\dfrac{1}{k}\ln\left(\dfrac{g}{g + ku_2}\right) = \dfrac{1}{k}\ln\left(1 + \dfrac{ku_2}{g}\right)$	A1	any equivalent form
	$y = \int \dot{y}\,dt = -\dfrac{1}{k^2}(g + ku_2)\,e^{-kt} - \dfrac{1}{k}gt + c_2$	M1	integrating
	$t = 0, y = 0 \Rightarrow c_2 = \dfrac{1}{k^2}(g + ku_2)$	M1	initial condition
	$y_{\max} = -\dfrac{1}{k^2}(g + ku_2)\dfrac{g}{g + ku_2} - \dfrac{1}{k}g\dfrac{1}{k}\ln\left(1 + \dfrac{ku_2}{g}\right) + \dfrac{1}{k^2}(g + ku_2)$	M1	substitute their t
	$= \dfrac{u_2}{k} - \dfrac{g}{k^2}\ln\left(1 + \dfrac{ku_2}{g}\right)$	E1	
	for unresisted motion, max. height $= \dfrac{u_2^{\,2}}{2g}$	B1	from constant acceleration formulae
	$\ln\left(1 + \dfrac{ku_2}{g}\right) \approx \dfrac{ku_2}{g} - \tfrac{1}{2}\left(\dfrac{ku_2}{g}\right)^2 + \dots$	M1	
	$y_{\max} \approx \dfrac{u_2}{k} - \dfrac{g}{k^2}\left(\dfrac{ku_2}{g} - \tfrac{1}{2}\left(\dfrac{ku_2}{g}\right)^2 + \dots\right)$	M1	
	$= \dfrac{u_2^{\,2}}{2g} +$ terms of order k	E1	needs B1
			10

2(i)		M1	diagram	
		A1	completely correct including angles	
	$\frac{V}{\sin 15°} = \frac{8}{\sin 135°}$	M1		
	$V = 2.93\,\text{m s}^{-1}$	A1		
	$\frac{v_b}{\sin 30°} = \frac{8}{\sin 135°}$	M1		
	$v_b = 4\sqrt{2} \approx 5.66\ \text{m s}^{-1}$	A1		6
(ii)		B1	diagram	
	$\frac{V}{\sin(\alpha - 45)°} = \frac{8}{\sin 45°}$	M1		
	$\Rightarrow \sin(\alpha - 45)° = \frac{V \sin 45°}{8} (= \sin 15°) \Rightarrow \alpha = 60$, so angle of 60°	A1		3
(iii)	${}_{\text{b}}\mathbf{v}_{\text{w}} = -8\cos 30°\mathbf{i} + 8\sin 30°\mathbf{j}$	B1		
	${}_{\text{c}}\mathbf{v}_{\text{w}} = U\mathbf{i}$	B1		
	${}_{\text{b}}\mathbf{v}_{\text{c}} = {}_{\text{b}}\mathbf{v}_{\text{w}} - {}_{\text{c}}\mathbf{v}_{\text{w}} = -\left(U + 4\sqrt{3}\right)\mathbf{i} + 4\mathbf{j}$	E1		
	collide if ${}_{\text{b}}\mathbf{v}_{\text{c}} \,//\, -100\mathbf{i} + 25\mathbf{j} \Rightarrow U = 16 - 4\sqrt{3}$	B1		4
(iv)	${}_{\text{b}}\mathbf{r}_{\text{c}} = (-a\mathbf{i} + 4\mathbf{j})t + 100\mathbf{i} - 25\mathbf{j}$	M1 A1		
	$\Rightarrow d^2 = (100 - at)^2 + (4t - 25)^2$	M1		
	$= (a^2 + 16)t^2 - 200(a+1)t + 10625$	E1		
	$d^2 > 5^2 \Leftrightarrow (a^2 + 16)t^2 - 200(a+1)t + 10600 > 0$	M1		
	true for all $t \Leftrightarrow [200(a+1)]^2 - 4(a^2 + 16)\cdot 10600 < 0$	M1		
	$\Leftrightarrow -3a^2 + 100a - 798 < 0$			
	$a < \frac{1}{3}\left(50 - \sqrt{106}\right)$ or $a > \frac{1}{3}\left(50 + \sqrt{106}\right)$			
	$U < 6.31$ or $U > 13.2$	A1		7

3(i)	$\mathbf{r} = r\,\hat{\mathbf{r}}$	M1		
	$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}})$	M1		
	$= \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$	E1		
	$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dot{r}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}}) + \ddot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{\theta}}) + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}}$	M1		
		A1		
	$= \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + \ddot{r}\,\hat{\mathbf{r}} - r\dot{\theta}^2\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}}$			
	$= (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}$	E1		
	$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} \Rightarrow$ transverse cpt. $= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})$	E1		
				7
(ii)	force // $\mathbf{r} \Rightarrow$ transverse cpt. = 0	M1		
	$\Rightarrow \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$	M1		
	$\Rightarrow r^2\dot{\theta} = \text{constant}$	A1		
				3
(iii)	$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{r^2}$	M1	allow $\frac{k}{r^3}$	
	$\ddot{r} - r\left(\frac{h}{r^2}\right)^2 = -\frac{k}{r^2}$	M1	substitute $\dot{\theta} = \frac{h}{r^2}$	
	$\ddot{r} = \frac{h^2}{r^3} - \frac{k}{r^2}$	A1		
	$\dot{r}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r} = \frac{h^2}{r^3} - \frac{k}{r^2}$	M1	writing $\ddot{r} = \dot{r}\frac{\mathrm{d}\dot{r}}{\mathrm{d}r}$	
	$\frac{1}{2}\dot{r}^2 = -\frac{h^2}{2r^2} + \frac{k}{r} + c$	M1	integrating	
	$\dot{r}^2 = \frac{2kr - h^2}{r^2} + A$	E1		
				6
(iv)	$\lvert\mathbf{v}\rvert^2 = (\dot{r})^2 + (r\dot{\theta})^2$	M1		
	$= \frac{2kr - h^2}{r^2} + A + r^2\left(\frac{h}{r^2}\right)^2$	M1		
	$= \frac{2k}{r} + A$			
	$\lvert\mathbf{v}\rvert = \sqrt{\frac{2k}{r} + A}$	A1		
	so speed maximised when r minimised, i.e. at closest approach	A1		
				4

4(i)	$\rho = k(a-r)$	B1		
	$M = \int_0^a k(a-r)2\pi r\,\mathrm{d}r$	M1	integral for mass; allow kr for $k(a-r)$	
	$= 2\pi k\left[\frac{1}{2}ar^2 - \frac{1}{3}r^3\right]_0^a$	M1		
	$= \frac{1}{3}\pi k a^3$	A1		
	$I = \int_0^a k(a-r)\cdot 2\pi r \cdot r^2\,\mathrm{d}r$	M1 A1	integral for I; allow kr for $k(a-r)$	
	$= 2\pi k\left[\frac{1}{4}ar^4 - \frac{1}{5}r^5\right]_0^a$	M1	integrating	
	$= \frac{1}{10}\pi k a^5 = \frac{3}{10}Ma^2$	E1		8
(ii)	perp.axes: $I_x + I_y = \frac{3}{10}Ma^2$	M1		
	symmetry $\Rightarrow I_x = I_y \Rightarrow I_x = \frac{3}{20}Ma^2$	A1		
	par.axes (G at origin) $I_{\text{tangent}} = I_x + Ma^2$	M1		
	$= \frac{23}{20}Ma^2$	E1		4
(iii)	$\frac{1}{2}I\omega^2 - Mga = 0$	M1 A1	energy correct equation	
	$\frac{1}{2}\cdot\frac{23}{20}Ma^2\omega^2 = Mga$	M1	solve for ω	
	$\omega = \sqrt{\frac{40g}{23a}} \approx \frac{4.13}{\sqrt{a}}$	A1		4
(iv)	$(\frac{23}{20}Ma^2)\ddot{\theta} = -Mga\sin\theta$	M1 A1	N2L	
	$\ddot{\theta} \approx -\frac{20g}{23a}\theta$	M1	using $\sin\theta \approx \theta$	
	$T \approx 2\pi\sqrt{\frac{23a}{20g}} \approx 2.15\sqrt{a}$	A1		4

Examiner’s Report

2611 Mechanics 5

General Comments

The entry of 89 for this year was well up on recent years and was very welcome and pleasing. There did however appear to be a significant number of candidates who seemed ill at ease with the paper and as a consequence perhaps did not do themselves justice. In contrast though there were many very good scripts and candidates should be pleased with their efforts.

I have noted on numerous occasions before that there is a problem with algebraic manipulation. Last year this did not do too much harm, but regrettably this time it did to a few candidates.

Comments on Individual Questions

Q.1 This question on the two dimensional motion through resisting air of a ball was on the whole reasonably well done. However, very few candidates were able to write down the *vector* equation of motion at the start in part (i). By contrast the majority could and did write down the *scalar* equations of motion and then for the most part were able to solve them! The responses from then on were only marred by algebraic errors in manipulation.

Q.2 There was a very sharp deterioration in the response this year to this relative velocity question. Although there were some quite good answers, for the most part the candidates seemed to have little idea how to proceed. It is true to say that historically there has been a tendency for candidates to have trouble with this topic, but in recent years this seemed to have improved somewhat. It is not easy to analyse the problem as there was really insufficient material to judge, but I would say that the absence or weakness of a diagram was at the heart of most difficulties. It is absolutely crucial in my view to express the information on a well-labelled diagram. From then on the problem often solves itself.

In later parts of the question, some candidates resorted to an algebraic approach with some success in a few cases. This alternative approach can be used in any relative velocity situation and for some people is a preferred approach. Consistency and accuracy are the keys though whatever method is adopted.

Q.3 I was very surprised with the response here. Apart from the final part perhaps, I expected that candidates would be entirely familiar with this sort of problem. The establishment of the equations of motion for a satellite motion moving under the effect of an inverse square law of central attraction is the entire basis of this topic. Candidates however gave the clear impression that they were not at all familiar with the material. Very few candidates indeed were able for instance to derive the vector velocity components from the position vector, although a similar request for the acceleration vector was found easy. In a way this is entirely in keeping with the failing noted above at the beginning of question 1. A very common error in part (ii) was to use an inverse cube law rather than the given inverse square. This is because candidates forgot that the position vector **r** has a modulus of r and not 1. The integration of the vector acceleration to find the velocity was either omitted or poorly done. Many though were able to have a stab at the final part using the given expression for the speed.

Q.4 Candidates were much happier with the first part of this question on moments of inertia. There were many good answers establishing the moments of inertia about various axes for a uniform circular disc. The final parts to use this information in a moving situation were not so good though. Part (iii) required the use of energy to find an angular speed whereas part (iv) needed an equation of motion to find a period of small oscillations. Candidates often interchanged the two applications for these two cases and ended up rather confusing themselves. I think that most knew what the theory was but were unable to apply it successfully.